

Hyperconvex representations and exponential growth

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Abstract

Let G be a real algebraic semi-simple Lie group and Γ be the fundamental group of a compact negatively curved manifold. In this article we study the limit cone, introduced by Benoist[2], and the growth indicator function, introduced by Quint[15], for a class of representations $\rho : \Gamma \rightarrow G$ admitting an equivariant map from $\partial\Gamma$ to the Furstenberg boundary of G 's symmetric space together with a transversality condition. We then study how these objects vary with the representation.

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1 Introduction

Consider a discrete subgroup of isometries Γ of a negatively curved space X . The exponential growth rate

$$\limsup_{s \rightarrow \infty} \frac{\log \#\{\gamma \in \Gamma : d(o, \gamma o) \leq s\}}{s}$$

plays a crucial role in understanding asymptotic properties of the group Γ : On nice situations this exponential growth rate coincides with the topological entropy of the geodesic flow on $\Gamma \backslash X$ on its non wandering set and with the Hausdorff dimension of Γ 's limit set on the visual boundary of X . Let us cite the work of Margulis[12], Patterson[13], Sullivan[22], just to name a few.

An important difference appears when one considers higher rank geometry. Let us briefly recall the work of Benoist[2] and Quint[15].

Consider G a connected real semi-simple algebraic group and consider some discrete subgroup Δ of G . Let K be a maximal compact subgroup of G τ the Cartan involution on \mathfrak{g} for which the set $\text{fix } \tau$ is K 's Lie algebra, consider $\mathfrak{p} = \{v \in \mathfrak{g} : \tau v = -v\}$ and \mathfrak{a} a maximal abelian subspace contained in \mathfrak{p} .

Let Σ be the roots of \mathfrak{a} on \mathfrak{g} , Σ^+ a system of positive roots on Σ and Π the set of simple roots associated to the choice Σ^+ . Let \mathfrak{a}^+ be a Weyl chamber and $a : G \rightarrow \mathfrak{a}^+$ the Cartan projection. Fix some norm $\|\cdot\|$ on \mathfrak{a} invariant under the Weyl group. If $\|\cdot\|$ is euclidean then $\|a(g)\|$ is the Riemannian distance on X ($= G$'s symmetric space) between $g \cdot o = g[K]$ and $o = [K]$, if $\|\cdot\|$ is not euclidean then $\|a(g)\|$ can be interpreted as $d(o, g \cdot o)$ for some G -invariant Finsler metric on X . The exponential growth rate one is interested in is

$$h_{\Delta}^{\|\cdot\|} := \limsup_{s \rightarrow \infty} \frac{\log \#\{g \in \Delta : \|a(g)\| \leq s\}}{s}.$$

Nevertheless, \mathfrak{a} being higher dimensional, one can consider the directions where the points $\{a(g) : g \in \Delta\}$ are. Benoist[2] has shown that the asymptotic cone of $\{a(g) : g \in \Delta\}$, i.e. the limit points of sequences $t_n a(g_n)$ where $t_n \in \mathbb{R}$ goes to zero and g_n belongs to Δ , coincides with the closed cone generated by the spectrum $\{\lambda(g) : g \in \Delta\}$, $\lambda : G \rightarrow \mathfrak{a}^+$ being the Jordan projection. One inclusion is trivial since

$$\frac{a(g^n)}{n} \rightarrow \lambda(g)$$

when $n \rightarrow \infty$ (c.f. Benoist[2]).

Theorem 1.1 (Benoist[2]). *Assume Δ is Zariski dense in G , then the asymptotic cone generated by $\{a(g) : g \in \Delta\}$ coincides with the closed cone generated by $\{\lambda(g) : g \in \Delta\}$. This cone is convex and has non empty interior.*

This cone is called the *limit cone* of Δ and denoted \mathcal{L}_{Δ} . Quint[15] is then interested in *how many* elements of $\{a(g) : g \in \Delta\}$ are in each direction of \mathcal{L}_{Δ} : Given an open cone $\mathcal{C} \subset \mathfrak{a}^+$ consider the exponential growth rate

$$h_{\mathcal{C}}^{\|\cdot\|} := \limsup_{s \rightarrow \infty} \frac{\log \#\{g \in \Delta : a(g) \in \mathcal{C} \text{ with } \|a(g)\| \leq s\}}{s},$$

the *growth indicator function*, introduced by Quint[15], is then the function $\psi_{\Delta} : \mathfrak{a} \rightarrow \mathbb{R} \cup \{-\infty\}$ defined as

$$\psi_{\Delta}(v) := \|v\| \inf\{h_{\mathcal{C}}^{\|\cdot\|} : \mathcal{C} \text{ open cone with } v \in \mathcal{C}\}.$$

One remarks that ψ_{Δ} is homogeneous and independent of the norm $\|\cdot\|$ chosen.

He shows the following theorem:

Theorem 1.2 (Quint[15]). *Let Δ be a Zariski dense discrete subgroup of G . Then ψ_{Δ} is concave and upper semi-continuous, the set*

$$\{v \in \mathfrak{a} : \psi_{\Delta}(v) > -\infty\}$$

is the limit cone \mathcal{L}_{Δ} of Δ . ψ_{Δ} is non negative on \mathcal{L}_{Δ} and positive on its interior.

Quint[15] shows that the exponential growth rate for a given norm $\|\cdot\|$ is then retrieved as

$$\sup_{v \in \mathfrak{a} - \{0\}} \frac{\psi_{\Delta}(v)}{\|v\|} = h_{\Delta}^{\|\cdot\|}.$$

This work consists in deeper study of these objects for hyperconvex representations. This notion has its origin at the work of Labourie[9].

Consider Γ a discrete co-compact torsion free isometry group of a negatively curved Hadamard manifold \widetilde{M} and denote $\mathcal{F} = G/P$ where P is a minimal parabolic subgroup of G . The space $\mathcal{F} \times \mathcal{F}$ has a unique open G -orbit denoted $\partial^2 \mathcal{F}$.

Definition 1.3. A representation $\rho : \Gamma \rightarrow G$ is *hyperconvex* if it admits a Hölder continuous ρ -equivariant map $\zeta : \partial\Gamma \rightarrow \mathcal{F}$ such that whenever $x, y \in \partial\Gamma$ are distinct the pair $(\zeta(x), \zeta(y))$ belongs to $\partial^2 \mathcal{F}$.

The main example of hyperconvex representation is the following: Consider Σ a closed orientable surface of genus $g \geq 2$ and say that a representation $\pi_1(\Sigma) \rightarrow \mathrm{PSL}(d, \mathbb{R})$ is *Fuchsian* if it factors as

$$\pi_1(\Sigma) \rightarrow \mathrm{PSL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(d, \mathbb{R})$$

where $\mathrm{PSL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(d, \mathbb{R})$ is the unique irreducible linear action $\mathrm{PSL}(2, \mathbb{R}) \curvearrowright \mathbb{R}^d$ (modulo conjugation by $\mathrm{PSL}(d, \mathbb{R})$) and $\pi_1(\Sigma) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ is co-compact. A *Hitchin component* of $\mathrm{PSL}(d, \mathbb{R})$ is a connected component of

$$\mathrm{hom}(\pi_1(\Sigma), \mathrm{PSL}(d, \mathbb{R})) = \{\text{morphisms } \rho : \pi_1(\Sigma) \rightarrow \mathrm{PSL}(d, \mathbb{R})\}$$

containing a Fuchsian representation.

Theorem (Labourie[9]). *A representation in a Hitchin component of $\mathrm{PSL}(d, \mathbb{R})$ is hyperconvex.*

In this work we begin by showing the following property of the limit cone of a hyperconvex representation:

Proposition (Corollary 3.12). *Consider $\rho : \Gamma \rightarrow G$ a Zariski dense hyperconvex representation, then the limit cone $\mathcal{L}_{\rho(\Gamma)}$ is contained in the interior of the Weyl chamber \mathfrak{a}^+ .*

Recall that \mathcal{L}_{Δ} is by definition closed, so the statement of the last proposition is stronger than “ $\lambda(\rho\gamma)$ belongs to the interior of the Weyl chamber for every $\gamma \in \Gamma$ ”.

The last proposition together with theorem C in A.S.[20] imply directly the following precise counting result. For g in $\mathrm{PGL}(d, \mathbb{R})$ denote $\lambda_1(g) \geq \lambda_2(g) \cdots \geq \lambda_d(g)$ the logarithm of the modulus of the eigenvalues (counted with multiplicity) of a lift $\tilde{g} \in \mathrm{GL}(d, \mathbb{R})$ of g , with determinant in $\{-1, 1\}$.

Corollary. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}(d, \mathbb{R})$ be a Zariski dense hyperconvex representation, and fix some $i \in \{1, \dots, d-1\}$, then there exists some positive $h = h_i$ such that*

$$hte^{ht} \#\{[\gamma] \in [\Gamma] : \lambda_i(\rho\gamma) - \lambda_{i+1}(\rho\gamma) \leq t\} \rightarrow 1$$

when $t \rightarrow \infty$, where $[\gamma]$ is the conjugacy class of γ .

Concerning the growth indicator function, we show the following theorem inspired in the work of Quint[17] for Schottky groups of G .

Theorem A (Corollary 4.9). *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense hyperconvex representation, then the growth indicator function $\psi_{\rho(\Gamma)} : \mathfrak{a} \rightarrow \mathbb{R}$ is strictly concave, analytic on the interior of $\mathcal{L}_{\rho(\Gamma)}$ and with vertical tangent on its boundary.*

Fix some hyperconvex representation $\rho : \Gamma \rightarrow G$ and denote ψ_ρ for its growth indicator function. Since ψ_ρ is strictly concave there exists a unique direction in the interior of the limit cone $\tau_\rho^\parallel \in \mathrm{int}(\mathcal{L}_\rho)$ such that the supremum of $\psi_\rho / \parallel \parallel$ is realized, this direction is called *growth direction* of $\rho(\Gamma)$ for the norm $\parallel \parallel$ (the uniqueness if this direction is also true for any Zariski dense subgroup Δ of G assuming the norm $\parallel \parallel$ is euclidean, which we shall not).

By definition the set of points in $\{a(\rho\gamma) : \gamma \in \Gamma\}$ outside a given open cone containing τ_ρ has exponential growth rate strictly smaller than h_ρ^\parallel .

In order to prove theorem A we use dual objects associated to \mathcal{L}_ρ and ψ_ρ : if a linear functional $\varphi \in \mathfrak{a}^*$ verifies $\varphi \geq \psi_\rho$ then

$$\|\varphi\| = \sup_{v: \|v\|=1} \varphi(v) \geq \sup_{\|v\|=1} \psi_\rho(v) = h_\rho^\parallel.$$

One is then led to consider the set

$$D_\rho = \{\varphi \in \mathfrak{a}^* : \varphi \geq \psi_\rho\}.$$

This set is a subset of the dual cone $\mathcal{L}_\rho^* = \{\varphi \in \mathfrak{a}^* : \varphi|_{\mathcal{L}_\rho} \geq 0\}$ since $\psi_\rho|_{\mathcal{L}_\rho} \geq 0$.

We then relate the set $D_{\rho(\Gamma)}$ with the thermodynamic formalism of the geodesic flow $\phi_t : \Gamma \backslash T^1 \widetilde{M} \rightarrow \Gamma \backslash T^1 \widetilde{M}$. This idea is already present in the work of Quint[17], nevertheless the way to find this relation is different and this method has the advantage of extending to (for example) Hitchin representations of surface groups.

We now briefly explain this relation:

Recall that periodic orbits of the geodesic flow $\phi_t : \Gamma \backslash T^1 \widetilde{M}$ are in correspondence with conjugation classes $[\gamma] \in [\Gamma]$, and recall that the pressure of some potential $f : \Gamma \backslash T^1 X \rightarrow \mathbb{R}$ is defined as

$$P(f) = \sup\{h(\phi_t, m) + \int f dm : m \text{ } \phi_t\text{-invariant probability}\}$$

where $h(\phi_t, m)$ is the metric entropy of ϕ_t with respect to the measure m . A probability maximizing $P(f)$ is called an equilibrium state of f . The equilibrium state of f is unique provided that f is Hölder continuous.

Following Quint[16] and Ledrappier[10] one finds a Γ -invariant Hölder continuous function $F_\rho : T^1\widetilde{M} \rightarrow \mathfrak{a}$ such that

$$\int_{[\gamma]} F_\rho = \lambda(\rho\gamma).$$

We then show:

Proposition (Proposition 4.7). *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense hyperconvex representation, then the set $D_{\rho(\Gamma)}$ is the set of functionals $\varphi \in \mathfrak{a}^*$ which are non negative on the limit cone such that $P(-\varphi(F_\rho)) \leq 0$.*

We then find the following nice dynamical interpretation of the growth indicator. For $\varphi \in \mathfrak{a}^*$ denote m_φ the equilibrium state of $\varphi(F_\rho) : \Gamma \backslash T^1\widetilde{M} \rightarrow \mathbb{R}$.

Corollary (Corollary 4.9). *Consider $\varphi_0 \in \mathfrak{a}^*$ tangent to ψ_ρ , then the direction where φ_0 and ψ_ρ are tangent is given by the vector $\int F_\rho dm_{\varphi_0}$ and the value of ψ_ρ in this vector is the metric entropy of the geodesic flow for the equilibrium state m_{φ_0}*

$$\psi_\rho\left(\int F_\rho dm_{\varphi_0}\right) = h(\phi_t, m_{\varphi_0}).$$

In the last section of this work we study continuity properties of these objects when the representation ρ varies.

Say that $\rho : \Gamma \rightarrow \mathrm{PGL}(d, \mathbb{R})$ is *strictly convex* if it is irreducible and admits two Hölder continuous ρ -equivariant maps $\xi : \partial\Gamma \rightarrow \mathbb{P}(\mathbb{R}^d)$ and $\eta : \partial\Gamma \rightarrow \mathbb{P}(\mathbb{R}^{d*})$ such that

$$\xi(x) \oplus \eta(y)$$

whenever x and y are distinct.

Proposition (Proposition 3.8). *The functions*

$$\rho \mapsto \limsup_{s \rightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) \leq s\}}{s}$$

and

$$\rho \mapsto \limsup_{s \rightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) - \lambda_d(\rho\gamma) \leq s\}}{s}$$

are continuous among strictly convex representations.

Consider a closed hyperbolic oriented surface Σ and denote $\mathrm{Hitchin}(\Sigma, d)$ the Hitchin components of the space

$$\mathrm{hom}(\pi_1(\Sigma), \mathrm{PGL}(d, \mathbb{R})) / \mathrm{PGL}(d, \mathbb{R}).$$

Since Hitchin representations are hyperconvex and irreducible (Labourie[9]) they are, in particular, strictly convex.

Corollary. *The function $\text{Hitchin}(\Sigma, d) \rightarrow \mathbb{R}$*

$$\rho \mapsto \lim_{s \rightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : (\lambda_1 - \lambda_d)(\rho\gamma) \leq s\}}{s}$$

is continuous.

This particular function is shown to be analytic by Pollicott-Sharp[14]. In fact, what stops us from obtaining more regularity, is that the equivariant map varies (only?) continuously with the representation. This is a consequence of the Anosov property, shown to hold by Guichard-Weinhard[8].

We return now to Zariski dense hyperconvex representations. We remark that the work of Guichard-Weinhard[8] implies that Zariski dense hyperconvex representations are an open set of the space of all representations $\Gamma \rightarrow G$.

We show in corollary 5.4 that the limit cone varies continuously with the representation. If one fixes a Zariski dense hyperconvex representation ρ and an open cone \mathcal{C} contained in the interior of \mathcal{L}_ρ , it will remain in the interior of the limit cone of all representations nearby. One can thus study the continuity of the growth indicator.

Theorem B (Theorem 5.6). *Let $\rho_0 : \Gamma \rightarrow G$ be a Zariski dense hyperconvex representation and fix some closed cone \mathcal{C} in the interior of the limit cone \mathcal{L}_{ρ_0} of ρ_0 . Consider some neighborhood U of ρ_0 such that \mathcal{C} is contained in $\text{int}(\mathcal{L}_\rho)$ for every $\rho \in U$, then the function $U \rightarrow \mathbb{R}$ given by*

$$\rho \mapsto \psi_\rho|_{\mathcal{C}}$$

is continuous.

We then find the following corollary:

Corollary (Corollary 5.8). *The function that associates to a Zariski dense hyperconvex representation ρ the exponential growth rate $h_{\rho(\Gamma)}^{\|\cdot\|}$ is continuous.*

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2 Anosov flows and Hölder cocycles

Reparametrizations

Let X be a compact metric space, $\phi_t : X \rightarrow X$ a continuous flow on X without fixed points and $f : X \rightarrow \mathbb{R}$ a positive continuous function. Set $\kappa : X \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$\kappa(x, t) = \int_0^t f\phi_s(x) ds, \quad (1)$$

if t is positive, and $\kappa(x, t) := -\kappa(\phi_t x, -t)$ for t negative. Thus, κ verifies the cocycle property $\kappa(x, t + s) = \kappa(\phi_t x, s) + \kappa(x, t)$ for every $t, s \in \mathbb{R}$ and $x \in X$.

Since $f > 0$ and X is compact f has a positive minimum and $\kappa(x, \cdot)$ is an increasing homeomorphism of \mathbb{R} . We then have an inverse $\alpha : X \times \mathbb{R} \rightarrow \mathbb{R}$ that verifies

$$\alpha(x, \kappa(x, t)) = \kappa(x, \alpha(x, t)) = t \quad (2)$$

for every $(x, t) \in X \times \mathbb{R}$.

Definition 2.1. The *reparametrization* of ϕ_t by f is the flow $\psi_t : X \rightarrow X$ defined as $\psi_t(x) := \phi_{\alpha(x, t)}(x)$. If f is Hölder continuous we shall say that ψ_t is a Hölder reparametrization of ϕ_t .

We say that some function $U : X \rightarrow \mathbb{R}$ is C^1 in the flow's direction if for every $p \in X$ the function $t \mapsto U(\phi_t(p))$ is of class C^1 and the function

$$p \mapsto \left. \frac{\partial}{\partial t} \right|_{t=0} U(\phi_t(p))$$

is continuous. Two Hölder potentials $f, g : X \rightarrow \mathbb{R}$ are then said to be *Livšic cohomologous* if there exists a continuous $U : X \rightarrow \mathbb{R}$, C^1 in the flow's direction, such that for all $p \in X$ one has

$$f(p) - g(p) = \left. \frac{\partial}{\partial t} \right|_{t=0} U(\phi_t(p)).$$

Remark 2.2. When two Hölder potentials $f, g : X \rightarrow \mathbb{R}_+^*$ are Livšic cohomologous the reparametrization of ϕ_t by f is conjugated to the reparametrization by g , i.e. there exists a homeomorphism $h : X \rightarrow X$ such that for all $p \in X$ and $t \in \mathbb{R}$

$$h(\psi_t^f p) = \psi_t^g(hp).$$

If m is a ϕ_t -invariant probability on X and ψ_t is the reparametrization of ϕ_t by f , then the probability m' defined by $dm'/dm(\cdot) = f(\cdot)/m(f)$ is ψ_t -invariant. In particular, if τ is a periodic orbit of ϕ_t then it is also periodic for ψ_t and the new period is

$$\int_{\tau} f = \int_0^{p(\tau)} f(\phi_s(x)) ds$$

for where $p(\tau)$ is the period of τ for ϕ_t and $x \in \tau$. This relation between invariant probabilities induces a bijection and Abramov[1] relates the corresponding metric entropies:

$$h(\psi_t, m') = h(\phi_t, m) / \int f dm. \quad (3)$$

Denote \mathcal{M}^{ϕ_t} the set of ϕ_t -invariant probabilities. The *pressure* of a continuous function $f : X \rightarrow \mathbb{R}$ is defined as

$$P(\phi_t, f) = \sup_{m \in \mathcal{M}^{\phi_t}} h(\phi_t, m) + \int_X f dm.$$

A probability m such that the supremum is attained is called an *equilibrium state* of f . An equilibrium state for the potential $f \equiv 0$ is called a probability with maximal entropy and its entropy is called the topological entropy of ϕ_t , denoted $h_{\text{top}}(\phi_t)$.

Lemma 2.3 (§2 of A.S.[20]). *Consider $\psi_t : X \curvearrowright$ the reparametrization of $\phi_t : X \curvearrowright$ by $f : X \rightarrow \mathbb{R}_+^*$, and assume that $h_{\text{top}}(\psi_t)$ is finite. Then the bijection $m \mapsto m'$ induces a bijection between equilibrium states of $-h_{\text{top}}(\psi_t)f$ and probabilities of maximal entropy of ψ_t .*

Anosov flows

Assume from now on that X is a compact manifold and that the flow $\phi_t : X \curvearrowright$ is C^1 . We say that ϕ_t is *Anosov* if the tangent bundle of X splits as a sum of three $d\phi_t$ -invariant bundles

$$TX = E^s \oplus E^0 \oplus E^u,$$

and there exist positive constants C and c such that: E^0 is the direction of the flow and for every $t \geq 0$ one has: for every $v \in E^s$

$$\|d\phi_t v\| \leq C e^{-ct} \|v\|,$$

and for every $v \in E^u$ $\|d\phi_{-t} v\| \leq C e^{-ct} \|v\|$.

One can compute the topological entropy of a reparametrization of an Anosov flow as the exponential growth rate of its periodic orbits.

Proposition 2.4 (Bowen[5]). *Let $\psi_t : X \curvearrowright$ be a reparametrization of an Anosov flow, then the topological entropy of ψ_t is*

$$h_{\text{top}}(\psi_t) = \limsup_{s \rightarrow \infty} \frac{\log \#\{\tau \text{ periodic} : p(\tau) \leq s\}}{s},$$

where $p(\tau)$ is the period of τ for ψ_t .

As shown by Bowen[6] transitive Anosov flows admit Markov partitions and thus the ergodic theory of suspension of sub shifts of finite type extends to this flows.

Proposition 2.5 (Bowen-Ruelle[7]). *Let $\phi_t : X \curvearrowright$ be a transitive Anosov flow. Then given a Hölder potential $f : X \rightarrow \mathbb{R}$ there exists a unique equilibrium state for f .*

Proposition 2.6 (cf. Ruelle[19]-Ratner[18]). *Let $\phi_t : X \curvearrowright$ be a transitive Anosov flow and $f, g : X \rightarrow \mathbb{R}$ be Hölder continuous. Then the function $t \mapsto P(f - tg)$ is analytic and*

$$\left. \frac{\partial P(f - tg)}{\partial t} \right|_{t=0} = - \int g dm_f$$

where m_f is f 's equilibrium state. If $\int g dm_f = 0$ and

$$\left. \frac{\partial^2 P(f - tg)}{\partial t^2} \right|_{t=0} = 0$$

then g is cohomologous to zero. Thus, if g is not cohomologically trivial and $\int g dm_f = 0$ then $t \mapsto P(f - tg)$ is strictly convex.

We will need the following lemma of Ledrappier[10].

Lemma 2.7 (Ledrappier[10], page 106). *Consider some potential $f : X \rightarrow \mathbb{R}$ such that $\int_\tau f \geq 0$ for every periodic orbit τ . If the number*

$$h := \limsup_{s \rightarrow \infty} \frac{\log \#\{\tau \text{ periodic} : \int_\tau f \leq s\}}{s}$$

belongs to $(0, \infty)$ then $P(-hf) = 0$. Conversely, if $P(-s_0 f) = 0$ for some $s_0 \in (0, \infty)$ then

$$s_0 = \limsup_{s \rightarrow \infty} \frac{\log \#\{\tau \text{ periodic} : \int_\tau f \leq s\}}{s} = h.$$

If this is the case

$$0 < \inf_{\tau \text{ periodic}} \frac{1}{p(\tau)} \int_\tau f \leq \sup_{\tau \text{ periodic}} \frac{1}{p(\tau)} \int_\tau f < \infty.$$

Hölder cocycles on $\partial\Gamma$

Denote Γ for a discrete co-compact torsion free isometry group, of a negatively curved complete simply connected manifold \widetilde{M} . Γ is then a hyperbolic group and its boundary $\partial\Gamma$ is naturally identified with \widetilde{M} 's visual boundary.

We will now focus on Hölder cocycles on $\partial\Gamma$.

Definition 2.8. A *Hölder cocycle* is a function $c : \Gamma \times \partial\Gamma \rightarrow \mathbb{R}$ such that

$$c(\gamma_0 \gamma_1, x) = c(\gamma_0, \gamma_1 x) + c(\gamma_1, x)$$

for any $\gamma_0, \gamma_1 \in \Gamma$ and $x \in \partial\Gamma$, and where $c(\gamma, \cdot)$ is a Hölder map for every $\gamma \in \Gamma$ (the same exponent is assumed for every $\gamma \in \Gamma$).

Given a Hölder cocycle c we define the *periods* of c as the numbers

$$\ell_c(\gamma) := c(\gamma, \gamma_+)$$

where γ_+ is the attractive fixed point of γ in $\Gamma - \{e\}$. The cocycle property implies that the period of an element γ only depends on its conjugacy class $[\gamma] \in [\Gamma]$.

The main result we shall use on Hölder cocycles is the following theorem of Ledrappier[10] which relates them to Hölder potentials on $T^1 \widetilde{M}$.

Theorem 2.9 (Ledrappier[10], page 105). *For each Hölder cocycle c there exists a Hölder continuous Γ -invariant function $F_c : T^1\widetilde{M} \rightarrow \mathbb{R}$ such that for every $\gamma \in \Gamma$ one has*

$$\ell_c(\gamma) = \int_{[\gamma]} F_c,$$

where $[\gamma]$ denotes the periodic orbit of the geodesic flow associated to γ .

Recall we have denoted $\int_{[\gamma]} F_c$ for the integral of F_c along the periodic orbit associated to γ .

One can find an explicit formula for such F_c as follows (Ledrappier[10] page 105): Fix some point $o \in \widetilde{M}$ and consider a C^∞ function $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support such that $f(0) = 1, f'(0) = f''(0) = 0$ and $f(t) > 1/2$ if $|t| \leq 2 \sup\{d(p, \Gamma \cdot o) : p \in \widetilde{M}\}$.

We can assume that $t \mapsto f(d(\phi_t(p, x)_b, q))$ is differentiable on t for every $p, q \in \widetilde{M}$, where $\phi_t(p, x)_b \in \widetilde{M}$ is the base point of $\phi_t(p, x) \in T^1\widetilde{M}$.

Set $A : \widetilde{M} \times \partial\Gamma \rightarrow \mathbb{R}$ to be

$$A(p, x) = \sum_{\gamma \in \Gamma} f(d(p, \gamma o)) e^{-c(\gamma^{-1}, x)}, \quad (4)$$

then the function $F_c : \widetilde{M} \times \partial\Gamma \rightarrow \mathbb{R}$

$$F_c(p, x) = - \left. \frac{d}{dt} \right|_{t=0} \log A(\phi_t(p, x)_b, x) \quad (5)$$

is Γ -invariant and verifies $\int_\gamma F_c = c(\gamma, \gamma_+)$.

From the explicit formula for F_c one can deduce some regularity properties.

Denote $\text{Holder}^\alpha(X)$ for the set of Hölder continuous real valued functions $f : X \rightarrow \mathbb{R}$ with exponent α where X is some compact metric space. For $f \in \text{Holder}^\alpha(X)$ denote $\|f\|_\infty := \max |f|$ and

$$K_f = \sup \frac{|f(p) - f(q)|}{d(p, q)^\alpha},$$

one then defines the norm $\|f\|_\alpha$ as $\|f\|_\alpha := \|f\|_\infty + K_f$. The vector space $(\text{Holder}^\alpha(X), \|\cdot\|_\alpha)$ is a Banach space.

If c is a Hölder cocycle with exponent α and $\gamma \in \Gamma$ define $\|c(\gamma, \cdot)\|_\alpha$ as its Hölder norm on $\text{Holder}^\alpha(\partial\Gamma)$.

Fix a finite generator \mathcal{A} of Γ and define the distance between two Hölder cocycles with same Hölder exponent c and c' as

$$d(c, c') := \sup\{\|c(\gamma, \cdot) - c'(\gamma, \cdot)\|_\alpha : \gamma \in \mathcal{A}\}.$$

Denote \mathcal{C}^α the vector space of Hölder cocycles with exponent α . It is clear that the topology of \mathcal{C}^α does not depend on the (finite) generator of Γ .

Corollary 2.10. *The function $c : \mathcal{C}^\alpha \rightarrow \text{Holder}^\alpha(T^1\widetilde{M})$*

$$c \mapsto F_c$$

given by formula (5) is analytic.

Proof. Consider a compact fundamental domain of Γ acting on \widetilde{M} and let W be a small neighborhood of this compact set. The set $\mathcal{A} = \{\gamma \in \Gamma : \gamma W \cap W \neq \emptyset\}$ is finite and a generator of Γ .

It is then clear that the function given by formula (4) $c \mapsto \log A(\cdot, \cdot)|_W \times \partial\Gamma$ is analytic since only the elements in $\gamma \in \mathcal{A}$ verify $\gamma W \cap W \neq \emptyset$ and \mathcal{A} is finite. An explicit formula for the derivative

$$t \mapsto - \left. \frac{d}{dt} \right|_{t=0} \log A(\phi_t(p, x)_b, x)$$

shows analyticity of $c \mapsto F_c|_W \times \partial\Gamma$. Since F_c is Γ -invariant and W contains a fundamental domain we obtain that $c \mapsto F_c$ is analytic. \square

Livšic[11]'s theorem implies that the set of Γ -invariant Hölder functions $F : T^1\widetilde{M} \rightarrow \mathbb{R}$ cohomologous to zero is a closed subspace of $\text{Holder}^\alpha(T^1\widetilde{M})$, one obtains thus the following:

Corollary 2.11. *The function $\mathcal{C}^\alpha \rightarrow \text{Holder}^\alpha(T^1\widetilde{M})/\{\text{Livšic cohomology}\}$*

$$c \mapsto \text{the cohomology class of } F_c,$$

is analytic.

We will always assume that the periods of a Hölder cocycle c are positive, i.e. $\ell_c(\gamma) > 0$ for every γ . For such a cocycle one defines the exponential growth rate as

$$h_c := \limsup_{s \rightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \ell_c(\gamma) \leq s\}}{s} \in (0, \infty]$$

(it is consequence of Ledrappier's theorem that h_c is always positive).

Lemma 2.12 (§2 of A.S.[20]). *Let $c : \Gamma \times \partial\Gamma \rightarrow \mathbb{R}$ be a Hölder cocycle with finite exponential growth rate, then the function F_c is cohomologous to a positive function and is not cohomologous to a constant.*

Denote $\text{Holder}_+^\alpha(T^1\widetilde{M})$ for the subset of $\text{Holder}^\alpha(T^1\widetilde{M})$ of functions cohomologous to a positive function.

Lemma 2.13. *The function $h : \text{Holder}_+^\alpha(T^1\widetilde{M}) \rightarrow \mathbb{R}$ given as a solution to the equation*

$$P(-h(F)F) = 0$$

is analytic. Moreover, the function $F \mapsto \text{equilibrium state of } -h(F)F$ is also analytic.

Proof. This is direct consequence of the implicit function theorem and of the formula

$$\left. \frac{\partial P(f - tg)}{\partial t} \right|_{t=0} = \int g dm_f$$

where m_f is f 's equilibrium state. \square

Denote \mathcal{C}_+^α the subset of Hölder cocycles with positive periods such that $h_c \in (0, \infty)$. We obtain the following proposition:

Proposition 2.14. *The exponential growth rate function $h : \mathcal{C}_+^\alpha \rightarrow \mathbb{R}$*

$$c \mapsto h_c$$

is analytic.

Proof. Consider some $c \in \mathcal{C}_+^\alpha$. Since h_c is finite and positive lemma 2.12 implies that the function F_c belongs to $\text{Holder}_+^\alpha(T^1\widetilde{M})$. One then applies corollary 2.10 together with lemma 2.13. \square

3 Convex representations

This section is devoted to the study of the limit cone of convex representations. We first work on strictly convex representations, i.e. irreducible morphisms $\rho : \Gamma \rightarrow \text{PGL}(d, \mathbb{R})$ admitting equivariant mappings to $\mathbb{P}(\mathbb{R}^d)$ and $\mathbb{P}(\mathbb{R}^{d*})$ with a transversality condition. We then use these representations to study Zariski dense hyperconvex representations.

Strictly convex representations

Recall that Γ is the fundamental group of compact negatively curved manifold. Fix some finite dimensional real vector space V .

Definition 3.1. We shall say that an irreducible representation $\rho : \Gamma \rightarrow \text{PGL}(V)$ is *strictly convex* if there exist two Hölder ρ -equivariant mappings $\xi : \partial\Gamma \rightarrow \mathbb{P}(V)$ and $\eta : \partial\Gamma \rightarrow \mathbb{P}(V^*)$ such that for every distinct points $x \neq y$ on $\partial\Gamma$ the line $\xi(x)$ doesn't belong to the kernel of $\eta(y)$.

We say that $g \in \text{PGL}(V)$ is *proximal* if (any lift of g to $\text{GL}(V)$) has a unique complex eigenvalue of maximal modulus, and its generalized eigenspace is one dimensional. This eigenvalue is necessarily real and its modulus is equal to $\exp \lambda_1(g)$. We will denote g_+ the g -fixed line of V consisting of eigenvectors of this eigenvalue and denote g_- the g -invariant complement of g_+ (this is $V = g_+ \oplus g_-$). g_+ is an attractor on $\mathbb{P}(V)$ for the action of g and g_- is a repelling hyperplane.

Lemma 3.2 (§5 of A.S.[20]). *Let $\rho : \Gamma \rightarrow \mathrm{PGL}(V)$ be a strictly convex representation. Then for every $\gamma \in \Gamma$ $\rho(\gamma)$ is proximal, $\xi(\gamma_+)$ is its attracting fixed line and $\ker \eta(\gamma_-)$ is the repelling hyperplane, where ξ and η are the ρ -equivariant maps of the definition.*

Fix some strictly convex representation $\rho : \Gamma \rightarrow \mathrm{PGL}(V)$. The choice of a norm $\| \cdot \|$ on V induces a Hölder cocycle $\beta_1 : \Gamma \times \partial\Gamma \rightarrow \mathbb{R}$ defined as

$$\beta_1(\gamma, x) = \log \frac{\|\rho(\gamma)v\|}{\|v\|}$$

where v belongs to the line $\xi(x)$. We remark that lemma 3.2 implies that the period $\beta_1(\gamma, \gamma_+)$ is exactly $\lambda_1(\rho\gamma)$, the logarithm of the spectral radius of $\rho\gamma$.

The following proposition is key in this work, it states that the cocycle β_1 has finite exponential growth rate.

Proposition 3.3 (§5 of A.S.[20]). *Let $\rho : \Gamma \rightarrow \mathrm{PGL}(V)$ be strictly convex, then the cocycle β_1 has finite exponential growth rate, this is*

$$\limsup_{s \rightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) \leq s\}}{s} < \infty$$

where $[\gamma]$ is the conjugacy class of γ in Γ .

Let \mathfrak{v} be the Cartan algebra

$$\mathfrak{v} = \{(w_1, \dots, w_d) \in \mathbb{R}^d : w_1 + \dots + w_d = 0\}$$

of $\mathrm{PGL}(d, \mathbb{R})$. We will show that the limit cone of a strictly convex representation doesn't intersect the walls $\{w \in \mathfrak{v}^+ : w_1 = w_2\}$ and $\{w \in \mathfrak{v}^+ : w_{d-1} = w_d\}$. The following lemma is from Benoist[4].

Lemma 3.4 (Benoist[4]). *Let $g \in \mathrm{PGL}(V)$ be proximal and let $V_{\lambda_2(g)}$ be the sum of the characteristic spaces of g whose eigenvalue is of module $\exp \lambda_2(g)$. Then for every $v \notin \mathbb{P}(g_-)$ with non zero component in $V_{\lambda_2(g)}$ one has*

$$\lim_{n \rightarrow \infty} \frac{\log d_{\mathbb{P}}(g^n(v), g_+)}{n} = \lambda_2(g) - \lambda_1(g).$$

Proof. Consider $u \in g_+$ and a the eigenvalue of u . By definition one has $\lambda_1(g) = \log |a|$. We consider then $T : g_- \rightarrow \mathbb{P}(\mathbb{R}^d)$ as $Tw = \mathbb{R}(w + u)$. T identifies the hyperplane g_- to the complement of $\mathbb{P}(g_-)$ in $\mathbb{P}(V)$. The action of g on $\mathbb{P}(V)$ is read, via this identification, as $\hat{g} : g_- \rightarrow g_-$

$$\hat{g}(w) = \frac{1}{a}gw.$$

One then finds, with a linear algebra argument, that

$$\frac{1}{n} \log \frac{\|g^n w\|}{|a|^n} \rightarrow \lambda_2(g) - \lambda_1(g)$$

for every $w \in g_-$ that is not contained in the characteristic spaces of eigenvalue with module $< \exp \lambda_2(g)$. \square

Lemma 3.5 (cf. Yue[24]). *There exist two positive constants a and b such that for every $\gamma \in \Gamma$ and any point $x \in \partial\Gamma - \{\gamma_-\}$ one has*

$$-a|\gamma| \leq \lim_{n \rightarrow \infty} \frac{\log d_o(\gamma^n x, \gamma_+)}{n} \leq -b|\gamma|.$$

One obtains the following corollary:

Corollary 3.6. *Let $\rho : \Gamma \rightarrow \mathrm{PGL}(d, \mathbb{R})$ be strictly convex, then there exists $k > 0$ such that for any $\gamma \in \Gamma$ one has*

$$\frac{\lambda_1 \rho(\gamma) - \lambda_2 \rho(\gamma)}{\lambda_1(\rho\gamma)} > k.$$

Consequently the limit cone of $\rho(\Gamma)$ doesn't intersect the walls $\{w \in \mathfrak{v}^+ : w_1 = w_2\}$ and $\{w \in \mathfrak{v}^+ : w_{d-1} = w_d\}$.

Proof. Since $\rho(\gamma)$ is proximal and its attractive line is $\xi(\gamma_+)$ one finds, after lemma 3.4 and that fact that ρ is irreducible, that

$$\lim_{n \rightarrow \infty} \frac{\log d_{\mathbb{P}}(\rho(\gamma)^n \xi(x), \xi(\gamma_+))}{n} = \lambda_2 \rho(\gamma) - \lambda_1 \rho(\gamma)$$

for a point $x \in \partial\Gamma - \{\gamma_-\}$. The fact that ξ is Hölder then implies that

$$\begin{aligned} \lambda_2 \rho(\gamma) - \lambda_1 \rho(\gamma) &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \log C d(\gamma^n x, \gamma_+)^{\kappa} \\ &= \lim_{n \rightarrow \infty} \kappa \frac{1}{n} \log d_o(\gamma^n x, \gamma_+). \end{aligned}$$

Lemma 3.5 implies that this quantity is smaller than $-\kappa b|\gamma|$. In order to finish the proof we need to compare $|\gamma|$ with $\lambda_1(\rho\gamma)$ for which we apply proposition 3.3 together with Ledrappier[10]'s lemma 2.7 to the cocycle β_1 :

$$0 < \frac{1}{m} < \inf_{[\gamma]} \frac{\lambda_1(\rho(\gamma))}{|\gamma|} \leq \sup_{[\gamma]} \frac{\lambda_1(\rho(\gamma))}{|\gamma|} < m$$

for some constant $m > 1$. □

Proposition 3.7 (Proposition 4.10 of Guichard-Weinhard[8] + proposition 2.1 of Labourie[9]). *The function that associates to a strictly convex representation its equivariant maps is continuous and the Hölder exponent of the equivariant maps can be chosen locally constant.*

We are now able to prove the following proposition. Recall that $\lambda_d(g)$ is the logarithm of the modulus of g 's smallest eigenvalue.

Proposition 3.8. *The functions*

$$\rho \mapsto h_1(\rho) := \limsup_{s \rightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) \leq s\}}{s}$$

and

$$\rho \mapsto h_{1d}(\rho) := \limsup_{s \rightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) - \lambda_d(\rho\gamma) \leq s\}}{s}$$

are continuous among strictly convex representations.

Proof. Since the cocycle β_1 has finite exponential growth rate, proposition 3.7 together with corollary 2.14 imply directly the continuity of $h_1(\rho)$.

We focus then on $h_{1d}(\rho)$. The dual representation $\rho^* : \Gamma \rightarrow \mathrm{PGL}(\mathbb{R}^{d*})$ given by $\rho^*(\gamma)\varphi = \varphi \circ \rho(\gamma^{-1})$ is also strictly convex. The cocycle associated to ρ^* ,

$$\beta_d(\gamma, x) = \log \frac{\|\rho^*(\gamma)\varphi\|}{\|\varphi\|}$$

where $\varphi \in \eta(x)$, has periods

$$\beta_d(\gamma, \gamma_+) = \lambda_1(\rho\gamma^{-1}) = -\lambda_d(\rho\gamma).$$

Consider now the Hölder cocycle $\beta_{1d} : \Gamma \times \partial\Gamma \rightarrow \mathbb{R}$ defined as

$$\beta_{1d}(\gamma, x) = \beta_1(\gamma, x) + \beta_d(\gamma, x).$$

The periods of β_{1d} are

$$\beta_{1d}(\gamma, \gamma_+) = \lambda_1(\rho\gamma) + \lambda_1(\rho\gamma^{-1}) = \lambda_1(\rho\gamma) - \lambda_d(\rho\gamma) > 0$$

for every $\gamma \in \Gamma$. Again by proposition 3.7 and corollary 2.14 it is sufficient to prove that the cocycle β_{1d} has finite exponential growth rate, but this is clear from the inequality

$$\lambda_1(g) - \lambda_d(g) \geq \lambda_1(g)$$

for every $g \in \mathrm{PGL}(d, \mathbb{R})$ together with the fact that $h_1(\rho)$ is finite. This finishes the proof. \square

Convex representations on some flag space

Strictly convex representations are then used to study Zariski dense representations $\Gamma \rightarrow G$ which have equivariant maps to G/P where P is some parabolic subgroup of G :

Consider a real semi-simple algebraic group G . Let K be a maximal compact subgroup of G τ the Cartan involution on \mathfrak{g} for which the set $\mathrm{fix} \tau$ is K 's Lie algebra, consider $\mathfrak{p} = \{v \in \mathfrak{g} : \tau v = -v\}$ and \mathfrak{a} a maximal abelian subspace contained in \mathfrak{p} .

Let Σ be the roots of \mathfrak{a} on \mathfrak{g} , Σ^+ a system of positive roots on Σ and Π the set of simple roots associated to the choice Σ^+ . To each subset θ of Π one associates a parabolic subgroup P_θ of G whose Lie algebra is, by definition,

$$\mathfrak{p}_\theta = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \langle \Pi - \theta \rangle} \mathfrak{g}_{-\alpha}$$

where $\langle \theta \rangle$ is the set of positive roots generated by θ and

$$\mathfrak{g}_\alpha = \{w \in \mathfrak{g} : [v, w] = \alpha(v)w \ \forall v \in \mathfrak{a}\}.$$

Every parabolic subgroup of G is conjugated to a unique P_θ .

Set W to be the Weyl group of Σ and note $u_0 : \mathfrak{a} \rightarrow \mathfrak{a}$ the biggest element in W , u_0 is the unique element in W that sends \mathfrak{a}^+ to $-\mathfrak{a}^+$. The *opposition involution* $i : \mathfrak{a} \rightarrow \mathfrak{a}$ is defined as $i := -u_0$.

Fix from now on $\theta \subset \Pi$ a subset of simple roots of G and write \mathcal{F}_θ for G/P_θ . We consider also $P_{i(\theta)}$, the parabolic group associated to

$$i(\theta) := \{\alpha \circ i : \alpha \in \theta\}.$$

The set $\mathcal{F}_{i(\theta)} \times \mathcal{F}_\theta$ possesses a unique open G -orbit, which we will denote $\partial^2 \mathcal{F}_\theta$.

Definition 3.9. We shall say that a representation $\rho : \Gamma \rightarrow G$ is θ -convex if there exist two ρ -equivariant Hölder maps $\xi : \partial\Gamma \rightarrow \mathcal{F}_\theta$ and $\eta : \partial\Gamma \rightarrow \mathcal{F}_{i\theta}$ such that if $x \neq y$ are distinct points in $\partial\Gamma$ then the pair $(\eta(y), \xi(x))$ belongs to $\partial^2 \mathcal{F}_\theta$.

A Π -convex representation (i.e., when the set θ is the full set of simple roots and thus the parabolic group P_Π is minimal) is called *hyperconvex*. The set \mathcal{F}_Π is the Furstenberg boundary \mathcal{F} of G 's symmetric space.

Consider $\{\omega_\alpha\}_{\alpha \in \Pi}$ the set of fundamental weights of Π .

Proposition 3.10 (Tits[23]). *For each $\alpha \in \Pi$ there exists a finite dimensional proximal irreducible representation $\Lambda_\alpha : G \rightarrow \mathrm{PGL}(V_\alpha)$ such that the highest weight χ_α of Λ_α is an integer multiple of the fundamental weight ω_α .*

Fix some θ and consider some $\alpha \in \theta$. Consider also $\Lambda_\alpha : G \rightarrow \mathrm{PGL}((, \mathbb{R})V_\alpha)$ a representation given by Tits's proposition. Since Λ_α is proximal and $\alpha \in \theta$, one obtains an equivariant mapping $\xi_\alpha : \mathcal{F}_\theta \rightarrow \mathbb{P}(V_\alpha)$.

The highest weight of the dual representation $\Lambda_\alpha^* : G \rightarrow \mathbb{P}(V_\alpha^*)$ is $\chi_\alpha \circ i$, one thus obtains an equivariant mapping $\eta_\alpha : \mathcal{F}_{i\theta} \rightarrow \mathbb{P}(V_\alpha^*)$. Moreover, the pair $(x, y) \in \partial^2 \mathcal{F}_\theta$ verifies

$$\eta_\alpha(x) | \xi_\alpha(y) \neq 0.$$

One deduces the following remark:

Remark 3.11. If $\rho : \Gamma \rightarrow G$ is Zariski dense and θ -convex then the composition $\Lambda_\alpha \circ \rho : \Gamma \rightarrow \mathrm{PGL}(V_\alpha)$ (where Λ_α is Tits's representation for $\alpha \in \theta$) is strictly convex.

Remark 3.11 together with corollary 3.6 imply the following corollary:

Corollary 3.12. *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense θ -convex representation. Then the limit cone of $\mathcal{L}_{\rho(\Gamma)}$ of $\rho(\Gamma)$ does not intersect the walls $\{v \in \mathfrak{a} : \alpha(v) = 0\}$ for every $\alpha \in \theta \cup i\theta$.*

In particular, the limit cone of a Zariski dense hyperconvex representation is contained in the interior of the Weyl chamber \mathfrak{a}^+ .

Proof. As observed before, if $\alpha \in \theta$ the composition $\Lambda_\alpha \rho : \Gamma \rightarrow \mathrm{PGL}(V_\alpha)$ is strictly convex. Applying corollary 3.6 for the representation $\Lambda_\alpha \rho$ implies the existence of some $\kappa_\alpha > 0$ such that

$$\frac{\alpha(\lambda(\rho\gamma))}{\chi_\alpha(\lambda(\rho\gamma))} = \frac{\lambda_1(\Lambda_\alpha \rho\gamma) - \lambda_2(\Lambda_\alpha \rho\gamma)}{\lambda_1(\Lambda_\alpha \rho\gamma)} > \kappa_\alpha.$$

□

Busemann cocycle

We shall now focus on hyperconvex representations, i.e. $\rho : \Gamma \rightarrow G$ admits a Hölder continuous equivariant map $\zeta : \partial\Gamma \rightarrow \mathcal{F}$ such that the pair $(\zeta(x), z(y))$ belongs to $\partial^2 \mathcal{F}$ whenever $x \neq y$.

Given such a representation there is a natural Hölder (vector) cocycle on the boundary of Γ that appears for which we need *Buseman's cocycle* on G introduced by Quint[16]: The set \mathcal{F} is K -homogeneous with stabilizer M , where K is a maximal compact subgroup of G , one then defines $\sigma : G \times \mathcal{F} \rightarrow \mathfrak{a}$ to verify the following equation

$$gk = l \exp(\sigma(g, kM))n$$

following Iwasawa's decomposition of $G = Ke^{\mathfrak{a}}N$, where N is the unipotent radical of $P_\Pi = P$.

The cocycle one naturally associates to a hyperconvex representation is then $\beta^\rho : \Gamma \times \partial\Gamma \rightarrow \mathfrak{a}$ defined as

$$\beta^\rho(\gamma, x) = \sigma(\rho(\gamma), \zeta(x)).$$

Let $\lambda : G \rightarrow \mathfrak{a}^+$ be the Jordan projection. If there is no confusion we will omit the superscript of $\beta^\rho = \beta$.

Lemma 3.13 (§5 of A.S.[20]). *The periods of β are $\beta(\gamma, \gamma_+) = \lambda(\rho\gamma)$.*

We will consider linear functionals on the dual cone

$$\mathcal{L}_\Delta^* := \{\varphi \in \mathfrak{a}^* : \varphi|_{\mathcal{L}_\Delta} \geq 0\}.$$

Lemma 3.14 (§5 of A.S.[20]). *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense hyperconvex representation and consider some φ in the dual cone \mathcal{L}_ρ^* , then the Hölder cocycle $\varphi \circ \beta : \Gamma \times \partial\Gamma \rightarrow \mathbb{R}$ has finite exponential growth rate if and only if φ belongs to the interior of \mathcal{L}_ρ^* .*

For a hyperconvex representation $\rho : \Gamma \rightarrow G$ consider $F_\rho : T^1 \widetilde{M} \rightarrow \mathfrak{a}$ given by Ledrappier[10]'s theorem 2.9 for the cocycle $\beta : \Gamma \times \partial\Gamma \rightarrow \mathfrak{a}$. The following corollary is direct consequence of the last lemma and lemma 2.12.

Corollary 3.15. *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense hyperconvex representation and fix some φ in the interior of the dual cone \mathcal{L}_ρ^* . Then the function $\varphi(F_\rho) : T^1 \widetilde{M} \rightarrow \mathbb{R}$ is cohomologous to a positive function, and is not cohomologous to a constant.*

4 The growth indicator is strictly concave

We shall now consider the *growth indicator function* introduced by Quint[15]. Recall that $a : G \rightarrow \mathfrak{a}^+$ is the Cartan projection and fix some norm $\| \cdot \|$ on \mathfrak{a} invariant under the Weyl group.

Consider Δ a discrete Zariski dense subgroup of G . For an open cone \mathcal{C} on \mathfrak{a}^+ consider the exponential growth rate

$$h_{\mathcal{C}} := \limsup_{s \rightarrow \infty} \frac{\#\{g \in \Delta : a(g) \in \mathcal{C} \text{ and } \|a(g)\| \leq s\}}{s}.$$

One then sets $\psi_{\Delta} : \mathfrak{a}^+ \rightarrow \mathbb{R}$ as

$$\psi_{\Delta}(v) := \|v\| \inf_{\mathcal{C} \text{ open cone: } v \in \mathcal{C}} h_{\mathcal{C}}.$$

Remark that ψ_{Δ} is homogeneous and does not depend on the norm chosen.

Theorem 4.1 (Quint[15]). *The function ψ_{Δ} is concave and upper semicontinuous, positive on \mathcal{L}_{Δ} and strictly positive on its relative interior. The set $\{v \in \mathfrak{a} : \psi_{\Delta}(v) > -\infty\}$ coincides with the limit cone \mathcal{L}_{Δ} .*

We need the following lemma of Quint[15]:

Lemma 4.2 (Lemma 3.1.3 of Quint[15]). *Let Δ be a Zariski dense subgroup of G and consider $\varphi \in \mathfrak{a}^*$. If $\varphi(v) > \psi_{\Delta}(v)$ for every $v \in \mathfrak{a} - \{0\}$ then the Poincare series*

$$\sum_{g \in \Delta} e^{-\varphi(a(g))} < \infty.$$

If there exists v such that $\varphi(v) < \psi_{\Delta}(v)$ then

$$\sum_{g \in \Delta} e^{-\varphi(a(g))} = \infty.$$

Fix a Zariski dense hyperconvex representation $\rho : \Gamma \rightarrow G$ and denote ψ_{ρ} for its growth indicator function. If $\varphi \in \mathfrak{a}^*$ verifies $\varphi \geq \psi_{\rho}$ then

$$\|\varphi\| \geq \sup \frac{\psi_{\rho}(v)}{\|v\|} = h_{\Delta}.$$

One is then interested on the set

$$D_{\rho} := \{\varphi \in \mathfrak{a}^* : \varphi \geq \psi_{\rho}\}.$$

Since ψ_{ρ} is non negative on the limit cone \mathcal{L}_{ρ} the set D_{ρ} is contained in the dual cone \mathcal{L}_{ρ}^* . For $\varphi \in \mathcal{L}_{\rho}^*$ define

$$h_{\varphi} = \lim_{s \rightarrow \infty} \frac{\log \#\{\gamma \in \Gamma : \varphi(a(\rho\gamma)) \leq s\}}{s}.$$

Remark that h_φ is the critical exponent of the Poincare series

$$\sum_{\gamma \in \Gamma} e^{-\varphi(a(\rho\gamma))}.$$

Quint's lemma 4.2 implies the following characterization of the set D_ρ .

Lemma 4.3. *The interior of the set D_ρ is the set of $\varphi \in \mathcal{L}_\rho^*$ such that $h_\varphi < 1$ and its boundary coincides with the set of linear functionals such that $h_\varphi = 1$.*

We are now interested in showing that the growth indicator function ψ_ρ of $\rho(\Gamma)$ is strictly concave with vertical tangent on the boundary of the limit cone \mathcal{L}_ρ .

One (trivial) consequence of theorem C on A.S.[20] is the following corollary:

Corollary 4.4. *If $\varphi \in \mathcal{L}_\rho^*$ then the exponential growth rate of the Hölder cocycle $\varphi \circ \beta$ coincides with the exponential growth of $\{a(\rho\gamma) : \gamma \in \Gamma\}$ i.e.*

$$\begin{aligned} & \limsup_{s \rightarrow \infty} \frac{\log \# \{[\gamma] \in [\Gamma] : \varphi(\lambda(\rho\gamma)) \leq s\}}{s} \\ &= \limsup_{s \rightarrow \infty} \frac{\log \# \{\gamma \in \Gamma : \varphi(a(\rho\gamma)) \leq s\}}{s} = h_\varphi \end{aligned}$$

This corollary allows us to link the growth indicator function with the thermodynamic formalism on the geodesic flow on $\Gamma \backslash T^1 \widetilde{M}$. We will give a description of the set D_ρ by considering the pressure function of potentials on $\Gamma \backslash T^1 \widetilde{M}$. Fix from now on a Γ -invariant function $F_\rho : T^1 \widetilde{M} \rightarrow \mathfrak{a}$ given by Ledrappier[10]'s theorem 2.9 for the vector cocycle $\beta : \Gamma \times \partial\Gamma \rightarrow \mathfrak{a}$.

Proposition 4.5. *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense hyperconvex representation. Then $D_\rho = \{\varphi \in \mathfrak{a}^* : P(-\varphi \circ F_\rho) \leq 0\}$. The interior of D_ρ is then the set*

$$\{\varphi \in \mathfrak{a}^* : P(-\varphi \circ F_\rho) < 0\}.$$

Proof. Recall that, after corollary 4.4, for every $\varphi \in \mathcal{L}_\rho^*$ one has

$$\begin{aligned} h_\varphi &= \limsup_{s \rightarrow \infty} \frac{\log \# \{\gamma \in \Gamma : \varphi(a(\rho\gamma)) \leq s\}}{s} \\ &= \limsup_{s \rightarrow \infty} \frac{\log \# \{[\gamma] \in [\Gamma] : \varphi(\lambda(\rho\gamma)) \leq s\}}{s}. \end{aligned}$$

Recall also that after lemma 4.3 the interior of D_ρ is the set of linear functionals $\varphi \in \text{int } \mathcal{L}_\rho^*$ with $0 < h_\varphi < 1$.

Consider then a linear functional φ in the interior of D_ρ , this is, $\varphi(v) > \psi_\rho(v) \forall v \in \mathfrak{a} - \{0\}$. We want to show that $P(-\varphi \circ F_\rho) < 0$.

Quint[15]'s theorem 4.1 states that ψ_ρ is positive on the interior of the limit cone and thus φ belongs to the interior of the dual cone \mathcal{L}_ρ^* , this is $\varphi|_{\mathcal{L}_\rho - \{0\}} > 0$. Moreover one has $h_\varphi < 1$.

Corollary 3.15 implies that $\varphi \circ F_\rho$ is cohomologous to a strictly positive function and is not cohomologous to a constant, proposition 2.6 then implies that $t \mapsto P(-t\varphi(F_\rho))$ has strictly negative derivative. Thus

$$t \mapsto P(-t\varphi(F_\rho))$$

is strictly decreasing. Ledrappier[10]'s lemma 2.7 implies then $P(-h_\varphi\varphi(F_\rho)) = 0$. One then finds that $P(-\varphi(F_\rho)) < 0$ (see figure 1).

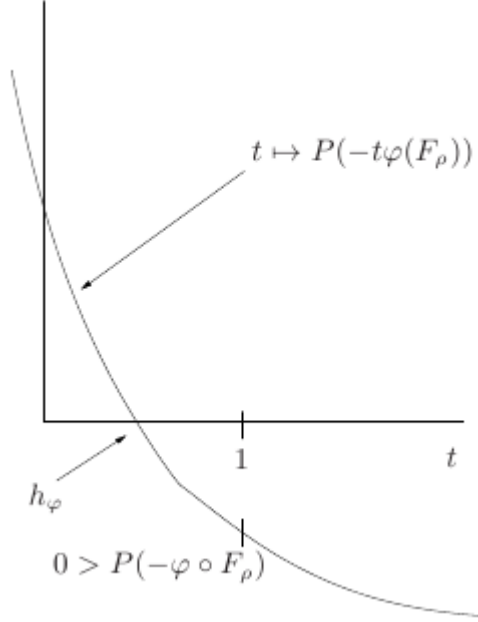


Figure 1: The function $t \mapsto P(-t\varphi(F_\rho))$ when $\varphi \circ F_\rho$ is cohomologous to a positive function and not cohomologous to a constant.

Conversely, fix a linear functional $\varphi \in \mathfrak{a}^*$ such that $P(-\varphi(F_\rho)) < 0$. Considering again the function $t \mapsto P(-t\varphi(F_\rho))$ one finds that, since $P(0) = h_{\text{top}}(\phi_t) > 0$ and $P(-\varphi(F_\rho)) < 0$, there exists some $0 < h < 1$ with $P(-h\varphi(F_\rho)) = 0$.

By definition of pressure one has

$$P(-\varphi(F_\rho)) = \sup_{m \in \mathcal{M}^{\phi_t}} h(m, \phi_t) - \int \varphi(F_\rho) dm < 0,$$

which implies that for every $\gamma \in \Gamma$

$$\int_{[\gamma]} \varphi(F_\rho) > 0,$$

this is $\varphi \circ \beta$ has positive periods. we can thus apply Ledrappier[10]'s lemma 2.7 and conclude that that such h is necessarily h_φ and thus φ belongs to the interior of D_ρ . \square

We will now deduce properties for ψ_ρ from properties of the pressure function. We need Benoist[3]'s theorem below:

Theorem 4.6 (Benoist[3]). *Consider Δ a Zariski dense subgroup of G , then the group generated by $\{\lambda(g) : g \in \Delta\}$ is dense in \mathfrak{a} .*

Proposition 4.7. *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense hyperconvex representation, then the set D_ρ is strictly convex and its boundary is an analytic sub manifold of \mathfrak{a}^* .*

Proof. Fix $F_\rho : T^1\widetilde{M} \rightarrow \mathfrak{a}$ given by Ledrappier[10]'s theorem 2.9 and consider the function $\overline{P} : \mathfrak{a}^* \rightarrow \mathbb{R}$ given by $\overline{P}(\varphi) = P(-\varphi(F_\rho))$. Proposition 2.6 implies that this function is analytic and its derivative $d\overline{P} : \mathfrak{a}^* \rightarrow \mathfrak{a}$ is given by the formula

$$d\overline{P}(\varphi) = - \int F_\rho dm_\varphi$$

where m_φ is the equilibrium state of $\varphi(F_\rho)$.

If $\varphi \in \mathfrak{a}^*$ is such that $\overline{P}(\varphi) = 0$ then proposition 4.5 implies that φ belongs to the boundary of the set D_ρ , in particular $\varphi \in \mathcal{L}_\rho^*$ and $h_\varphi = 1$. One deduces that $\varphi(F_\rho)$ is cohomologous to a positive function (corollary 3.15) and thus

$$\int \varphi(F_\rho) dm_\varphi \neq 0.$$

Hence the vector

$$d\overline{P}(\varphi) = \int F_\rho dm_\varphi \neq 0.$$

We conclude that 0 is a regular value of \overline{P} and thus $\partial D_\rho = \overline{P}^{-1}\{0\}$ is an analytic sub manifold of \mathfrak{a}^* .

The tangent space to ∂D_ρ at $\varphi_0 \in \partial D_\rho$ is

$$T_{\varphi_0} \partial D_\rho = \{\varphi \in \mathfrak{a}^* : \int \varphi(F_\rho) dm_{\varphi_0} = 0\}.$$

Consider then $\varphi \in T_{\varphi_0} \partial D_\rho$. Since the periods of F_ρ generate a dense subgroup of \mathfrak{a} (Benoist's theorem 4.6) the function $\varphi(F_\rho)$ is not cohomologous to zero. Proposition 2.6 then implies that the function

$$t \mapsto \overline{P}(\varphi_0 - t\varphi)$$

is strictly convex with one critical point at 0. Thus $\varphi_0 + T_{\varphi_0} \partial D_\rho$ does not intersect ∂D_ρ (except at φ_0) and thus D_ρ is strictly convex. \square

The following lemma is a consequence of Hahn-Banach's theorem, one can find a proof in §4.1 of Quint[17]:

Lemma 4.8 (Duality lemma). *Let V be a finite dimensional vector space and $\Psi : V \rightarrow \mathbb{R} \cup \{-\infty\}$ a concave homogeneous upper semi-continuous function. Set*

$$V_\Psi^* = \{\Phi \in V^* : \Phi \geq \Psi\} \text{ and } L_\Psi = \{x \in V : \Psi(x) > -\infty\}.$$

Suppose that V_Ψ^ and L_Ψ have non empty interior, then*

- *For every $x \in L_\Psi$ one has*

$$\Psi(x) = \inf_{\Phi \in V_\Psi^*} \Phi(x),$$

- *the set V_Ψ^* is strictly convex if and only if Ψ is differentiable on the interior of L_Ψ and with vertical tangent on the boundary.*
- *the boundary ∂V_Ψ^* is differentiable if and only if the function Ψ is strictly concave.*

When this conditions are satisfied, the derivative induces bijection between the set of directions contained in the interior of L_Ψ and ∂V_Ψ^ .*

We find the following corollary:

Corollary 4.9. *The growth indicator ψ_ρ of a Zariski dense hyperconvex representation ρ is strictly concave, analytic on the interior of the limit cone and with vertical tangent on its boundary. If $P(-\varphi_0(F_\rho)) = 0$ then φ_0 is tangent to ψ_ρ in the direction given by the vector*

$$\int F_\rho dm_{\varphi_0}$$

and the value

$$\psi_\rho\left(\int F_\rho dm_{\varphi_0}\right) = h(\phi_t, m_{\varphi_0})$$

is the metric entropy of the geodesic flow for the equilibrium state m_{φ_0} .

Proof. Proposition 4.7 together with the duality lemma 4.8 imply then that:

- i) Since D_ρ is strictly convex, ψ_ρ is of class C^1 on the interior of the cone of \mathcal{L}_ρ but with vertical tangent on its boundary,
- ii) Since ∂D_ρ is of class C^1 , ψ_ρ is strictly concave.

Proposition 2.6 implies that the derivative of the bijection between interior directions of \mathcal{L}_ρ and ∂D_ρ is invertible. Thus, since ∂D_ρ is analytic, the derivative of ψ_ρ is analytic on the interior of \mathcal{L}_ρ and thus ψ_ρ is analytic.

The formula

$$d\bar{P}(\varphi) = - \int F_\rho dm_\varphi$$

together with the first point of the duality lemma 4.8 imply that $\varphi_0 \in \partial D_\rho$ is tangent to ψ_ρ in the direction given by the vector $\int F_\rho dm_{\varphi_0}$, moreover one has

$$\varphi_0\left(\int F_\rho dm_{\varphi_0}\right) = \psi_\rho\left(\int F_\rho dm_{\varphi_0}\right).$$

Since $\varphi_0(F_\rho)$ is cohomologous to a positive function (corollary 3.15), we can consider $\sigma_t : \Gamma \backslash T^1 \widetilde{M} \hookrightarrow$ the reparametrization of the geodesic flow by $\varphi_0(F_\rho)$. Applying lemma 2.3 and Abramov[1]'s formula (3) we have that the topological entropy of σ_t verifies

$$h_{\text{top}}(\sigma_t) = h(\phi_t, m_{\varphi_0}) / \int \varphi_0(F_\rho) dm_{\varphi_0}.$$

Following proposition 2.4 the topological entropy of σ_t is the exponential growth rate of its periodic orbits, i.e. $h_{\text{top}}(\sigma_t) = h_{\varphi_0}$, this last quantity is equal to 1 since $\varphi_0 \in \partial D_\rho$ and thus

$$\psi_\rho\left(\int F_\rho dm_{\varphi_0}\right) = \int \varphi_0(F_\rho) dm_{\varphi_0} = h(\phi_t, m_{\varphi_0}).$$

This finishes the proof □

5 Continuity properties

In this section we are interested in showing that the objects one associates to a Zariski dense θ -representation vary continuously with the representation. We are concerned in the cone \mathcal{L}_ρ , the growth indicator ψ_ρ and the growth form Θ_ρ .

For a Zariski dense hyperconvex representation $\rho : \partial \Gamma \rightarrow G$ denote $F_\rho : T^1 \widetilde{M} \rightarrow \mathfrak{a}$ the function given by Ledrappier[10]'s theorem 2.9 for the cocycle β^ρ (this choice is only valid modulo Livsic cohomology).

Recall that $\phi_t : \Gamma \backslash T^1 \widetilde{M} \hookrightarrow$ is the geodesic flow and denote \mathcal{M}^{ϕ_t} the set of ϕ_t -invariant probabilities (for all t).

Lemma 5.1. *The set*

$$\left\{ \int F_\rho dm : m \in \mathcal{M}^{\phi_t} \right\}$$

is compact, doesn't contain $\{0\}$ and generates the cone \mathcal{L}_ρ^θ .

Proof. Compactness is immediate since \mathcal{M}^{ϕ_t} is compact. Considering some φ in the interior of the dual cone \mathcal{L}_ρ^* and applying lemma 3.14 together with Ledrappier's lemma 2.7 one obtains that zero does not belong to $\{\int F_\rho dm : m \in \mathcal{M}^{\phi_t}\}$.

The limit cone \mathcal{L}_ρ is the closed cone generated by the spectrum $\lambda(\rho\gamma) = \int_{[\gamma]} F_\rho$. Since convex combination of periodic orbits are dense in \mathcal{M}^{ϕ_t} (Anosov's closing lemma c.f. Shub[21]) the last statement follows. □

Denote $\text{hom}_{\Pi}^Z(\Gamma, G)$ for the set of Zariski dense hyperconvex representations endowed with the topology as subset of $G^{\mathcal{A}}$ where \mathcal{A} is a finite generator of Γ .

Guichard-Weinhard[8] have shown that Zariski dense hyperconvex representations are the so called Anosov representations, and are thus an open set in the space of representations. From this result together with Labourie[9] one obtains the following proposition:

Proposition 5.2 (Theorem 4.11 of Guichard-Weinhard[8] + proposition 2.1 of Labourie[9]). *The function that associates to a Zariski dense hyperconvex representation its equivariant maps is continuous and the Hölder exponent of the equivariant maps can be chosen locally constant.*

This proposition together with corollary 2.10 give:

Proposition 5.3. *The function that associates to every $\rho \in \text{hom}_{\Pi}^Z(\Gamma, G)$ the cohomology class of $F_{\rho} : T^1\widetilde{M} \rightarrow \mathfrak{a}$ is continuous.*

One directly obtains the continuity of the cone \mathcal{L}_{ρ} .

Corollary 5.4. *The function $\text{hom}_{\Pi}^Z(\Gamma, G) \rightarrow \{\text{compact subsets of } \mathbb{P}(\mathbb{R}^d)\}$ given by $\rho \mapsto \mathbb{P}(\mathcal{L}_{\rho})$ is continuous.*

Proof. Obvious from lemma 5.1 and proposition 5.3. \square

Corollary 5.5. *Fix some Zariski dense hyperconvex representation ρ_0 and consider some φ in the interior of the dual cone $\mathcal{L}_{\rho_0}^*$. Then the function*

$$\rho \mapsto h_{\varphi}(\rho) := \lim_{s \rightarrow \infty} \frac{\log \# \{[\gamma] \in [\Gamma] : \varphi(\lambda(\rho\gamma)) \leq s\}}{s}$$

is continuous in a neighborhood U of ρ_0 such that $\varphi|_{\mathcal{L}_{\rho}} - \{0\} > 0$ for $\rho \in U$.

Proof. Since the equivariant maps vary continuously with the representation, the function $\rho \mapsto \varphi \circ \beta^{\rho} : \Gamma \times \partial\Gamma \rightarrow \mathbb{R}$ is continuous. The exponential growth rate of $\varphi \circ \beta^{\rho}$ is $h_{\varphi}(\rho)$ and thus the corollary is consequence of corollary 2.14. \square

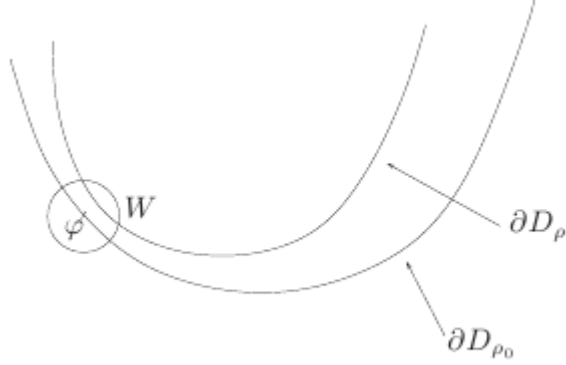
We can now show theorem B.

Theorem 5.6. *Let $\rho_0 : \Gamma \rightarrow G$ be a Zariski dense hyperconvex representation and fix some open cone \mathcal{C} such that its closure is contained in the interior of the limit cone \mathcal{L}_{ρ_0} of ρ_0 . Consider a neighborhood U of ρ_0 such that \mathcal{C} is contained in $\text{int}(\mathcal{L}_{\rho})$ for every $\rho \in U$, then the function $U \rightarrow \mathbb{R}$ given by*

$$\rho \mapsto \psi_{\rho}|_{\mathcal{C}}$$

is continuous.

Proof. Consider the set D_{ρ_0} of linear functionals $\varphi \in \mathfrak{a}^*$ such that $\varphi \geq \psi_{\rho_0}$, and more precisely its boundary ∂D_{ρ_0} . It suffices to prove that for some fixed $\varphi \in \partial D_{\rho_0}$ and a given neighborhood W of φ there exists a neighborhood U of ρ_0 such that for every $\rho \in U$ one has that ∂D_{ρ} intersects W .

Figure 2: The set D_ρ after perturbation.

Proposition 4.5 states that ∂D_ρ is the set of linear functionals in \mathcal{L}_ρ^* such that $P(-\varphi(F_\rho)) = 0$ lemma 2.13 together with proposition 5.3 imply thus the result. \square

Strict concavity of ψ_ρ together with the last theorem imply the following corollaries. Fix some norm $\|\cdot\|$ on \mathfrak{a} invariant under the Weyl group.

Corollary 5.7. *The function that $\Theta_\rho^{\|\cdot\|} : \text{hom}_\Pi^Z(\Gamma, G) \rightarrow \mathfrak{a}^*$ that associates to ρ the growth form $\Theta_\rho^{\|\cdot\|}$ is continuous.*

Proof. Recall that $\Theta_\rho^{\|\cdot\|}$ is the functional tangent to ψ_ρ in the direction that $\psi_\rho(\cdot)/\|\cdot\|$ attains its maximum. Since ψ_ρ is strictly concave, this direction belongs to the interior of \mathcal{L}_ρ . This remark together with theorem 5.6 imply the result. \square

Recall we have denoted $h_\rho^{\|\cdot\|}$ for the exponential growth rate

$$h_\rho^{\|\cdot\|} = \lim_{s \rightarrow \infty} \frac{\log \#\{\gamma \in \Gamma : \|a(\rho\gamma)\| \leq s\}}{s}.$$

Corollary 5.8. *The function $\rho \mapsto h_\rho^{\|\cdot\|} : \text{hom}_\Pi(\Gamma, G) \rightarrow \mathbb{R}$*

$$\rho \mapsto h_{\rho(\Gamma)}^{\|\cdot\|}$$

is continuous.

Proof. Obvious since $h_\rho^{\|\cdot\|}$ coincides with the norm of the growth form,

$$\|\Theta_\rho^{\|\cdot\|}\| = h_{\rho(\Gamma)}^{\|\cdot\|},$$

the result then follows from the last corollary. \square

References

- [1] L.M. Abramov. On the entropy of a flow. *Dokl. Akad. Nauk. SSSR*, 128, 1959.
- [2] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. funct. anal.*, 7(1), 1997.
- [3] Y. Benoist. Propriétés asymptotiques des groupes linéaires ii. *Adv. Stud. Pute Math.*, 26, 2000.
- [4] Y. Benoist. Convexes divisibles I. In *Algebraic groups and arithmetic*, pages 339–374. Tata Inst. Fund. Res., 2004.
- [5] R. Bowen. Periodic orbits of hyperbolic flows. *Amer. J. Math.*, 94, 1972.
- [6] R. Bowen. Symbolic dynamics for hyperbolic flows. *Amer. J. Math.*, 95, 1973.
- [7] R. Bowen and D. Ruelle. The ergodic theory of axiom A flows. *Invent. Math.*, 29, 1975.
- [8] O. Guichard and A. Weinhard. Anosov representations: domains of discontinuity and applications. ArXiv: 1108.0733v1, 2011.
- [9] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165, 2006.
- [10] F. Ledrappier. Structure au bord des variétés à courbure négative. *Séminaire de théorie spectrale et géométrie de Grenoble*, 71, 1994-1995.
- [11] A.O. Lopes and Ph. Thieullen. Sub-actions for Anosov flows. *Ergod. Th. & Dynam. Sys.*, 25, 2005.
- [12] G. Margulis. Applications of ergodic theory to the investigation of manifolds with negative curvature. *Functional Anal. Appl.*, 3, 1969.
- [13] S.-J. Patterson. The limit set of a fuchsian group. *Acta mathematica*, 136, 1976.
- [14] M. Pollicott and R. Sharp. Length asymptotics in higher teichmüller theory. Preprint.
- [15] J.-F. Quint. Divergence exponentielle des sous-groupes discrets en rang supérieur. *Comment. Math. Helv.*, 77, 2002.
- [16] J.-F. Quint. Mesures de Patterson-Sullivan en rang supérieur. *Geom. funct. anal.*, 12, 2002.
- [17] J.-F. Quint. L'indicateur de croissance des groupes de Schottky. *Ergod. Th. & Dynam. Sys.*, 23, 2003.

- [18] M. Ratner. The central limit theorem for geodesic flows on n -manifolds of negative curvature. *Israel J. Math.*, 16, 1973.
- [19] D. Ruelle. *Thermodynamic Formalism*. Addison-Wesley, London, 1978.
- [20] A. Sambarino. Quantitative properties of convex representations. Submitted. arXiv:1104.4705v1, 2011.
- [21] M. Shub. *Global stability of dynamical systems*. Springer Verlag, 1987.
- [22] D. Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Publ. Math. de l'I.H.E.S.*, 50, 1979.
- [23] J. Tits. Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque. *J. Reine Angew. Math.*, 247, 1971.
- [24] C. Yue. The ergodic theory of discrete isometry groups on manifolds of variable negative curvature. *Trans. of the A.M.S.*, 348(12), 1996.

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